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ON FINITE MOUFANG POLYGONS

BY

H. VAN MALDEGHEM*
Seminar for Geometry and Combinatorics
State University of Ghent
Krijgslaan 281
9000 Gent, Belgium

AND

RICHARD WEISS**
Department of Mathematics
Tufts University
Medford, MA 02155, U.S.A.

ABSTRACT

We show that a finite generalized polygon Γ is Moufang with respect to a group G if and only if for every flag $\{x, y\}$ of Γ, the subgroup $G_1(x, y)$ of G fixing every element incident with one of x, y acts transitively on the set of apartments containing the elements u, x, y, w, where $u \neq y$ (resp. $w \neq x$) is an arbitrary element incident with x (resp. y).

1. Introduction

Let Γ be an arbitrary undirected graph and let G be a subgroup of $\mathrm{aut}(\Gamma)$. For each vertex x, we denote by $\Gamma(x)$ the set of vertices adjacent to x and by $G(x)$ the stabilizer of x in G. For each $i \in \mathbb{N}$, we set

$$G_i(x) = \bigcap_{\{u \mid \partial(x,u) \leq i\}} G(u),$$

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where $\partial(x,u)$ denotes the distance from x to u. A *k-path* (for $k \in \mathbb{N}$) is an $(k+1)$-tuple $(x_0,\ldots,x_k)$ of vertices such that $x_i \in \Gamma(x_{i-1})$ if $1 \le i \le k$ and $x_i \ne x_{i-2}$ if $2 \le i \le k$. Let

$$G(x_0,\ldots,x_k) = G(x_0)\cap\cdots\cap G(x_k) \text{ and } G_i(x_0,\ldots,x_k) = G_i(x_0)\cap\cdots\cap G_i(x_k)$$

for each k-path $(x_0,\ldots,x_k)$ and each $i \in \mathbb{N}$. An *m-circuit* (for $m \ge 3$) in Γ is an m-path $(x_0,\ldots,x_m)$ with $x_0 = x_m$.

As in [14], we call a graph Γ a *generalized n-gon* (for $n \ge 2$) if every two edges of Γ lie on a common $2n$-circuit of Γ and Γ contains no m-circuits for $m < 2n$. The $2n$-circuits of Γ are called *apartments*. Let $|\Gamma(x)| = s+1$ and $|\Gamma(y)| = t+1$ for some edge $\{x,y\}$. If s and $t \ge 2$, then s and t are easily seen to be independent of the edge $\{x,y\}$; in this case, Γ is called *thick*. The pair (s,t) is called the *order* of Γ. For an introduction to generalized polygons and related concepts, see [6], [11] and [13].

A generalized polygon n-gon with $n \ge 3$ is called *Moufang* with respect to some $G \le \mathrm{aut}(\Gamma)$ if for every n-path $(x_0,\ldots,x_n)$, the subgroup $G_1(x_1,\ldots,x_{n-1})$ acts transitively on the set of apartments of Γ through $(x_0,\ldots,x_n)$. In [15] and several as yet unpublished papers, Tits has classified Moufang polygons. They are the rank 2 spherical buildings associated to certain classical or algebraic simple groups (for $n = 3$, 4 and 6), to certain so-called mixed groups of type G_2 (for $n = 6$) and to the Ree groups of type ${}^2F_4$ (for $n = 8$). The first step [14] in the classification of Moufang polygons is to show that $n = 3$, 4, 6 or 8. An essential ingredient in the proof (see [19]) is the following simple fact, a special case of [12, (4.1.1)]:

(1.1): *If Γ is a generalized n-gon and $G \le \mathrm{aut}(\Gamma)$, then for each edge $\{x,y\}$, the stabilizer in $G_1(x,y)$ of an apartment through $\{x,y\}$ is trivial.*

Now let $3 \le k \le n$ and suppose Γ is a generalized n-gon. We will say that Γ is *k-Moufang* with respect to some $G \le \mathrm{aut}(\Gamma)$ if for each k-path $(x_0,\ldots,x_k)$, the subgroup $G_1(x_1,\ldots,x_{k-1})$ acts transitively on the set of apartments of Γ through $(x_0,\ldots,x_k)$. By definition, Γ is Moufang with respect to G whenever it is n-Moufang with respect to G. It is easy to deduce from (1.1) (see (7.1) below) that Γ must, in fact, be Moufang with respect to G whenever it is 4-Moufang with respect to G. Our main result is the following:

(1.2) THEOREM: *Let Γ be a finite thick generalized n-gon with $n \geq 3$ which is 3-Moufang with respect to some $G \leq \mathrm{aut}(\Gamma)$. Then Γ is Moufang with respect to G.*

Note that the case $n = 4$ is also proved in [16] in a more geometrical fashion.

In [2], Fong and Seitz classified finite n-gons Γ with $n \geq 3$ having a group $G \leq \mathrm{aut}(\Gamma)$ such that for every edge $\{x, y\}$ of Γ, the stabilizer $G(x, y)$ contains a normal, nilpotent subgroup acting transitively on the set of apartments through $\{x, y\}$. It is well known that the Moufang property implies this hypothesis; an easy proof of this implication based only on [18] is contained in (3.1)–(3.2) below.

We do not know if (1.2) holds for infinite generalized polygons.

2. Preliminary observations

Our proof of (1.2) will depend on the following results which are descended from earlier results of Thompson [10], Wielandt [20] and Gardiner [3]: Let Γ be an arbitrary connected, undirected graph and G a subgroup of $\mathrm{aut}(\Gamma)$ such that for each vertex x, the stabilizer $G(x)$ acts primitively on $\Gamma(x)$ (so, in particular, G acts transitively on the edge set of Γ). Suppose, too, that $G_2(x) \neq G_1(x, y)$ for each vertex x. Then there exists a vertex x and a prime p such that for each $y \in \Gamma(x)$, the following assertions hold:

(2.1) $O_p(G(x, y)) \not\leq G_1(y)$.

(2.2) $G_1(x, y)/G_2(y)$ *is a p-group.*

(2.3) *If $O_p(G(x, y)) \not\leq G_1(x)$, then $G_1(x, y)$ is a p-group.*

Proof: These facts follow from the arguments of [18]. For the convenience of the reader, we give the details here. Choose an edge $\{x, y\}$ and let $H = G(x, y)$. We show first that every minimal normal subgroup of H lies in $G_1(x)$. Let A be such a subgroup and suppose that $A \not\leq G_1(x)$. Choose $w \in \Gamma(x)$ and let $a \in A \backslash G(w)$. Since $G_1(x) \trianglelefteq H$, we have $A \cap G_1(x) = 1$ and thus $[A, G_1(x)] = 1$. It follows that $G_1(w, x) = G_1(w, x)^a = G_1(x, w^a)$. But this implies that $G_1(w, x) \trianglelefteq \langle G(w, x), G(w^a, x)\rangle = G(x)$ and hence $G_1(w, x) \leq G_2(x)$, which contradicts our assumption. By a similar argument, every minimal normal subgroup of H lies in $G_1(y)$ as well.

We show next that $\mathrm{soc}(H) \leq \mathrm{soc}(G_1(x))$, where $\mathrm{soc}(X)$ denotes the subgroup generated by all the minimal normal subgroups of a group X. Again, let A be a minimal normal subgroup of H. Then $A \trianglelefteq G_1(x)$; let B be a minimal normal subgroup of $G_1(x)$ contained in A. Let $B^* = \langle B^h \mid h \in H\rangle$. Then $B^* \leq A$ and $B^* \trianglelefteq H$; thus $B^* = A$. Since each B^h is a minimal normal subgroup of $G_1(x)$, it follows that $A = B^* \leq \mathrm{soc}(G_1(x))$. Thus $\mathrm{soc}(H) \leq \mathrm{soc}(G_1(x))$ as claimed. By a similar argument, $\mathrm{soc}(H) \leq \mathrm{soc}(G_1(y))$.

If $\mathrm{soc}(G_1(x)) = \mathrm{soc}(H) = \mathrm{soc}(G_1(y))$, then $\mathrm{soc}(H) \trianglelefteq \langle G(x), G(y)\rangle$. Since Γ is connected, $\langle G(x), G(y)\rangle$ acts transitively on the edge set of Γ. This implies, however, that $\mathrm{soc}(H) = 1$ and therefore $H = 1$. We conclude that $\mathrm{soc}(H) \not\geq \mathrm{soc}(G_1(u))$ for $u = x$ or y. Now let B be a minimal normal subgroup of $G_1(u)$ not contained in $\mathrm{soc}(H)$ and A a minimal normal subgroup of H contained in $B^* = \langle B^h \mid h \in H\rangle$. Since B is a minimal normal subgroup of $G_1(u)$, either $A \cap B = 1$ or $B \leq A$. Since, however, $A \leq \mathrm{soc}(H)$ and $B \not\leq \mathrm{soc}(H)$, we must have $A \cap B = 1$ and thus $[A, B] = 1$. Hence $[A, B^h] = [A, B]^h = 1$ for all $h \in H$, and so $[A, B^*] = 1$. Since $A \leq B^*$, it follows that A is abelian. Thus $O_P(H) \neq 1$ for some prime p.

Let $K = G_1(x, y)$. If $O_p(H) \leq K$, then $O_p(G_1(x)) = O_p(H) = O_p(G_1(y))$, which implies that $O_p(H) \trianglelefteq \langle G(x), G(y)\rangle$. Since $O_p(H) \neq 1$, we can choose x and y so that (2.1) holds. Choose $z \in \Gamma(y)$ and an element $a \in O_p(H)\backslash G(z)$. Let $M = O^p(G_1(y, z))$ and $N = M^a$. Then $N \trianglelefteq G_1(y)$ and MN/N is a subgroup of $(MO_p(H) \cap G_1(y))/N$. Since $|MO_p(H)|/|N|$ is a power of p, so is $|MN/N| = |M/M \cap N|$. Since $O^p(M) = M$, it follows that $M = N$. Thus $M \trianglelefteq \langle G(y, z), G(y, z^a)\rangle = G(y)$, so $O^p(K) = M \leq G_2(y)$, so (2.2) holds. If $O_p(H) \not\leq G_1(x)$ as well, then $O^p(K) \trianglelefteq G(x)$ by a similar argument, so $O^p(K) = 1$ and (2.3) follows. ∎

Now suppose that Γ is an arbitrary generalized n-gon. A subgraph Δ of Γ is called *convex* if for every $m < n$, Δ contains every m-path of Γ whose end-points lie in Δ. The following result is well known (see, for instance, [17]):

(2.4): *Let Γ be a generalized n-gon and Δ a convex subgraph containing an $(n+1)$-path $(x_0, \ldots, x_{n+1})$. Let $1 < k < n$ and suppose that $|\Delta(x_i)| \geq 3$ for $i = k-1$ and k. Then Δ is a sub-n-gon; moreover, Δ is the convex closure of $(x_0, \ldots, x_{n+1}) \cup \Delta(x_{k-1}) \cup \Delta(x_k)$ in Γ.*

Proof: Let Φ be the convex closure of $(x_0, \ldots, x_{n+1}) \cup \Delta(x_{k-1}) \cup \Delta(x_k)$ in Γ. Then $\Phi \subseteq \Delta$. Let x_i' (for $i = k-1$ and k) be the vertex opposite x_i on the unique apartment through $(x_0, \ldots, x_{n+1})$; these vertices lie in Φ. To each vertex in $\Gamma(x_k)$ there is a unique nearest vertex in $\Gamma(x'_k)$. Since $\Delta(x_k) = \Phi(x_k)$, it follows that $\Delta(x'_k) = \Phi(x'_k)$. Choose $v \in \Delta(x_{k-1})\backslash\{x_{k-2}, x_k\}$. Then x'_k and v are at distance n in Γ, so now $\Delta(x'_k) = \Phi(x'_k)$ implies that $\Delta(v) = \Phi(v)$. The subgraph Φ contains the unique apartment through v, x_{k-1}, x_k, x'_{k-1}. Since a convex subgraph of Γ is clearly connected, we conclude that $\Delta = \Phi$ and that $|\Delta(u)| = |\Delta(x_{k-1})|$ or $|\Delta(x_k)|$ for every vertex u of Δ. In particular, $|\Delta(u)| \geq 3$ for every vertex u of Δ, from which it follows that Δ is itself a generalized n-gon. ∎

3. The proof of (1.2): First part

Let Γ be a finite thick n-gon with $n \geq 4$ which is 3-Moufang with respect to some subgroup G of aut(Γ). (If $n = 3$, there is nothing to prove.) By [1], we have $n = 4$, 6 or 8. Then $G_2(x) \neq G_1(x, y)$ for each edge $\{x, y\}$ and $G(x)$ acts 2-transitively on $\Gamma(x)$ for each vertex x, so (2.1)–(2.3) hold for some vertex x and some prime p. Let $(x_0, x_1, \ldots, x_{2n})$ be a $2n$-circuit in Γ with $x_1 = x$. Let $|\Gamma(x_0)| = s + 1$ and $|\Gamma(x_1)| = t + 1$. By (2.1), s is a multiple of p and by (2.2), t is a power of p. By (1.1), $|G_1(x_0, x_1)| = (st)^{(n-2)/2}$, the number of $(n+1)$-paths of Γ beginning with a given 3-path.

(3.1): *If s is a power of p, then $O_p(G(x_0, x_1))$ acts transitively on the set of apartments through $\{x_0, x_1\}$ (so Γ is Moufang with respect to G by the theorem [2] of Fong and Seitz).*

Proof: If s is a power of p, so is $|G_1(u, v)|$ for each edge $\{u, v\}$ of Γ. Since $G_1(x_0, x_1)$, $G_1(w, x_0)$ and $G_1(x_1, z)$ are subnormal in $G(x_0, x_1)$ for all $w \in \Gamma(x_0)\backslash\{x_1\}$ and all $z \in \Gamma(x_1)\backslash\{x_0\}$, it follows that $O_p(G(x_0, x_1))$ acts transitively on the set of 3-paths $(u_0, \ldots, u_3)$ with $u_1 = x_0$ and $u_2 = x_1$ and hence on the set of apartments through $\{x_0, x_1\}$. ∎

Let $H = G_1(x_2, x_3) \cap G(x_1, \ldots, x_{n+1})$. Then $|H| = t$.

(3.2): *If $H = G_1(x_2, \ldots, x_n)$, then s is a power of p.*

Proof: We have $H \trianglelefteq G_1(x_3, \ldots, x_n) \trianglelefteq \cdots \trianglelefteq G_1(x_n) \trianglelefteq G(x_n, x_{n+1})$. Since H is a p-group, it follows that $H \leq O_p(G(x_n, x_{n+1}))$. By (1.1), $H \not\leq G_1(x_{n+1})$ and hence $O_p(G(x_n, x_{n+1})) \not\leq G_1(x_{n+1})$ as well. Thus (2.3) implies that s is a power of p. ∎

We will call a sub-n-gon of Γ *full* if it is thick and of order (s', t') with $s' = s$ or $t' = t$ but not both.

(3.3): *Let $m < k - 1 \leq m + n - 2$ and suppose*

$$a \in G_1(x_{k-1}) \cap G(x_m, \ldots, x_{m+n}) \backslash G_1(x_k)$$

has $r \geq 2$ fixed points in $\Gamma(x_k) \backslash \{x_{k-1}\}$. Then Γ contains a full sub-n-gon of order (r, t) if k is even and order (s, r) if k is odd.

Proof: The subgroup $G(x_m \ldots, x_{m+n}) \cap G_1(x_{k-1}, x_k)$ contains an element b which maps $(x_m, \ldots, x_{m+n}, x_{m+n+1})$ to $(x_m, \ldots, x_{m+n}, x^a_{m+n+1})$. Thus

$$ab \in G_1(x_{k-1}) \cap G(x_m, \ldots, x_{m+n+1}),$$

so the fixed points of ab form a full sub-n-gon of Γ by (2.4). ∎

(3.4): *If Γ contains no full sub-n-gons, then $H = G_1(x_2, \ldots, x_n)$.*

Proof: If $H \neq G_1(x_2, \ldots, x_n)$, then we can choose $k \leq n$ minimal such that $H \not\leq G_1(x_2, \ldots, x_k)$. Choose $a \in H \backslash G_1(x_2, \ldots, x_k)$ and let r be the number of fixed points of a in $\Gamma(x_k) \backslash \{x_{k-1}\}$. Since $|H| = t$ is a power of p and p divides s, we have $r \geq 2$. Thus (3.3) applies. ∎

By the results of this section, we can assume from now on that s is not a power of p and that Γ contains a full sub-n-gon Δ containing the apartment $(x_0, x_1, \ldots, x_{2n})$.

4. Subpolygons

We interrupt the proof of (1.2) a moment to record the results on full subpolygons we will need. Let Γ be an arbitrary thick finite generalized n-gon of order (k, m) and suppose that Γ contains a full sub-n-gon Δ of order (k', m).

(4.1): $n \neq 8$.

Proof: If $n = 8$, then $m = (k')^2$ and $k = m^2$ by the last theorem in [9]. This implies that $km = (k'm)^2$, which contradicts the fact that $2km$ must be a perfect square [1]. ∎

(4.2): *If $n = 4$ or 6, then $k' \le m$. If $k' = m$, then $k = m^{n/2}$.*

Proof: The case $n = 4$ is handled in [5,(2.2.2)]. If $n = 6$, then $km \ge (k'm)^2$ by [8]. Since $k \le m^3$ by [4], it follows that $k' \le m$ and $k = m^3$ if $k' = m$. ∎

We return now to the situation at the end of §3. By (4.1), $n = 4$ or 6. Let D denote the setwise stabilizer of Δ in G and $E \le D$ the pointwise stabilizer. Let $f = 0$ or 1 and choose $z \in \Delta(x_{n+f})\backslash\{x_{n+f-1}\}$. There exists an element $a \in G_1(x_{f+1}, x_{f+2}) \cap G(x_f, \dots, x_{n+f})$ mapping z to x_{n+f+1}. By (2.4), Δ is the unique sub-n-gon of Γ having the same order as Δ and containing $\Delta(x_{f+1}) \cup \Delta(x_{f+2}) \cup \{x_f, \dots, x_{n+f+1}\}$. It follows that $a \in D$. Thus Δ is 3-Moufang with respect to D/E. By induction, we can assume that Δ is Moufang with respect to D/E, so, by (3.1)–(3.2), [2] applies. In particular, the parameters of Δ are prime powers. Since s is not a prime power, (4.2) implies that the parameters of Δ are distinct and that $|\Delta(x_1)| - 1 = t$ is the larger. We conclude that Δ is a $U_4(q)$-quadrangle with $t = q^2$ or a $U_5(q)$-quadrangle with $t = q^3$ if $n = 4$ and Δ is a ${}^3D_4(q)$-hexagon with $t = q^3$ if $n = 6$.

5. The proof of (1.2): Conclusion

Let $M = G_1(x_4, x_5) \cap G(x_3, \dots, x_{n+3})$. Again by (2.4), $H = G_1(x_2, x_3) \cap G(x_1, \dots, x_{n+1})$ and M both lie in D. Suppose now that $n = 4$. Let $a \in H$ be arbitrary. By the root-group structure of $U_4(q)$ and $U_5(q)$ (see §6 below), we can choose a nontrivial element $b \in M$ such that $[a, b] \in E$. Then in fact $[a, b] \in G_1(x_3, x_4) \cap E$, so $[a, b] = 1$ by (1.1). Since $a \in G_1(x_2)$, it follows that $a \in G_1(x_2^b)$ as well. Since $b \ne 1$, we have $x_2 \ne x_2^b$ by (1.1), so the subgroup $G_1(x_1) \cap G(x_1, x_2, x_3)$ contains an element c mapping x_2^b to x_4. Then $a^c \in G_1(x_2, x_3, x_4)$ and $(c^{-1})^a \in G_1(x_1)$, so $[a, c] = a^{-1}a^c = (c^{-1})^a c \in G_1(x_1, x_2, x_3) \cap G(x_1, \dots, x_5) = 1$. Hence $a = a^c \in G_1(x_4)$, and we are done by (3.2).

It remains to consider the case $n = 6$. By the root-group structure of ${}^3D_4(q)$ (see §6 below), we can choose nontrivial elements $a \in H \le D$ and $b \in M$ such that $[a, b] \in E$. Then $[a, b] \in G_1(x_3, x_4) \cap E = 1$, so a lies in $G_1(x_2)$ and $G_1(x_2^b)$ but in neither $G_1(x_1)$ nor $G_1(x_1^b)$. Let $U = G_1(x_2, x_3, x_4) \cap D$. If $c \in D(x_2, x_3)$ maps x_2^b to x_4, then $a^c \in U$. Thus U acts nontrivially on both $\Gamma(x_1)$ and $\Gamma(x_5)$. The group U is normalized by $D(x_0, \dots, x_{2n})$, the inverse image in D of a Cartan subgroup of D/E. By the structure of D (see §6 below), this implies that U acts transitively on the set of apartments of Δ through $(x_1,\ldots,x_5)$. Thus $|G_1(x_2,x_3,x_4)\cap G(x_1,\ldots,x_7)|\geq t$, so $H=G_1(x_2,x_3,x_4)\cap G(x_1,\ldots,x_7)$. Then $H\leq G_1(x_5)$ follows by (3.3) since all sub-hexagons of Γ are of order (q,t).

Now let d be an arbitrary element of H and e an element of G mapping $(x_1,\ldots,x_7)$ to $(x_7,\ldots,x_1)$. Since H acts transitively on $\Gamma(x_1)\backslash\{x_2\}$, there exists an element $c\in H$ such that $dc^e\in G(x_8)$. Thus $dc^e\in G_1(x_3,x_4)\cap G(x_1,\ldots,x_8)=1$. Since $c^e\in G_1(x_6)$, it follows that $d\in G_1(x_6)$ and we are again done by (3.2).

6. Properties of $U_4(q)$, $U_5(q)$ and ${}^3D_4(q)$

We give here an explanation of the properties of $U_4(q)$, $U_5(q)$ and ${}^3D_4(q)$ used in the previous section. Let $F=D/E$; for each n-path $(u_0,\ldots,u_n)$ in Δ, let $F[u_0,\ldots,u_n]$ denote the corresponding root group in F and

$$F[u_m,\ldots,u_n]=\langle F[u_0,\ldots,u_n],F[u_1,\ldots,u_{n+1}],\ldots,F[u_m,\ldots,u_{m+n}]\rangle$$

for $1\leq m<n$. Let $F_k=F[x_k,\ldots,x_{k+n}]$ for $k=1$, 2 and 3. We have $H\cong F_1$, $M\cong F_3$ and $[F_1,F_3]\leq F_2$. Choose $a\in F_1$. Since $[F_2,F_3]=1$, the map ϕ: $F_3\to F_2$ given by $b^\phi=[a,b^{-1}]$ for $b\in F_3$ is a homomorphism. Since $|F_2|<|F_3|$, the kernel of ϕ is nontrivial.

Now let $n=6$ and $W=UE/E$. Then W is a subgroup of $F[x_1,\ldots,x_5]$ which acts nontrivially on both $\Gamma(x_1)$ and $\Gamma(x_5)$ and is normalized by $F(x_0,x_1,\ldots,x_{12})$. We claim that $W=F[x_1,\ldots,x_5]$. If $q=2$, this is easily verified by calculating with the presentation for ${}^3D_4(2)$ given in [7,(3.8)]. If $q>2$, then [7,(2.7)] implies that W induces a group of order q^6 on $\Gamma(x_1)\cup\Gamma(x_5)$; since $[F[x_{11},x_0,\ldots x_5],F_1]=F_0$, the claim follows.

7. 4-Moufang polygons

For the sake of completeness, we observe:

(7.1) THEOREM: *Let Γ be a thick generalized n-gon, finite or infinite, with $n\geq 4$ which is 4-Moufang with respect to some $G\leq \mathrm{aut}(\Gamma)$. Then Γ is Moufang with respect to G.*

Proof: Let $(x_0,\ldots,x_{n+1})$ be an $(n+1)$-path in Γ and let $k\leq n$ be maximal such that the subgroup $K=G_1(x_1,x_2,x_3)\cap G(x_0,\ldots,x_n)$ lies in $G_1(x_k)$. We need to show that $k=n-1$, so suppose that $k<n-1$ and choose $a\in K\backslash G_1(x_{k+1})$. There

exists $b \in G_1(x_{k-1}, x_k, x_{k+1})$ mapping $(x_0, \ldots, x_n, x_{n+1}^a)$ to $(x_0, \ldots, x_n, x_{n+1})$. Thus $ab \in G_1(x_{k-1}, x_k) \cap G(x_0, \ldots, x_{n+1}) = 1$ by (1.1). But this implies that $a \in G_1(x_{k+1})$. With this contradiction, we conclude that $k = n - 1$. ∎

We remark that our proof of (1.2) is derived from an earlier, more geometric proof of the first author. In that proof, one considers compositions (not always commutators!) of *flag-elations* (i. e. collineations belonging to $G_1(x_1, x_2)$ for some flag $\{x_1, x_2\}$) in such a way that the resulting collineation ϕ either fixes some subgeometry, or acts semi-regularly on a set of s or $s-1$ points (or t or $t-1$ lines). In the first case, one shows that these subgeometries are either Moufang full subpolygons or the flag-complex of a Moufang generalized polygon, hence one of the parameters of Γ is a prime power. In the latter case we have conditions on the order of ϕ and s or t and if we play it right, then eventually this leads to the prime power condition. Of course it is the interplay of the arguments in the two cases that makes it work. Concerning this geometrical proof, the first author whishes to thank F. Buekenhout and J. A. Thas for some helpful comments.

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